

Differential invariants of feedback transformations for quasi-harmonic oscillation equations

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Abstract

The classification problem for a control-parameter-dependent second-order differential equations is considered. The algebra of the differential invariants with respect to Lie pseudo-group of feedback transformations is calculated. The equivalence problem for a control-parameter-dependent quasi-harmonic oscillation equation is solved. Some canonical forms of this equation are constructed.

Keywords: quasi-harmonic oscillation equations, feedback transformations, jet spaces, infinitesimal transformations, Lie pseudo-groups, differential invariants, invariant differentiations, local equivalence

1. Introduction

The feedback transformations of the control-parameter-dependent systems are analogous to Lie transformations of the differential equations. They are often used for reduction of such systems to canonical forms and simplification of equations.

It is generally known that the classification problem for the ordinary differential equations (ODEs) with respect to Lie transformations is among the key ones in the theory of the differential equations. The important results in this area were obtained by Kruglikov *et al.* in [1]. At the same time the solutions of the classification problem for general systems of the control-parameter-dependent ODEs are practically absent. Usually the classification problems for control-parameter-dependent affine systems ([2, 3, 4]), i.e. the systems of the form

$$\dot{x} = A(x) + B(x)u$$

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are considered. The classification problem for scalar equations of the form

$$\frac{dy}{dx} = f(x, y, u)$$

with respect to general feedback transformations was solved by V. V. Lychagin [5, 6]. The local equivalence problem for scalar control systems under the feedback pseudo-group was solved in [7] applying the method of moving frames. The classification problem for Hamilton systems with only one degree of freedom and scalar control parameter was solved in [8] by A. G. Kushner and V. V. Lychagin. The generalization of the problem for Hamilton systems with many degrees of freedom was given in [9].

In this paper the solutions of the classification and equivalence problems for QHO equations are presented.

Consider the following differential equation:

$$\frac{d^2y}{dx^2} + f(y, u) = 0, \quad (1)$$

where the function $f(y, u)$ is smooth. Here u is a scalar control parameter. This equation has a variety of applications. We will call an equation of form (1) *control-parameter-dependent quasi-harmonic oscillator equation* (QHO).

Consider the problems of equivalence and classification for such an equation with respect to the feedback transformations [8]:

$$\varphi : (x, y, u) \mapsto (X(x, y), Y(x, y), U(x, y, u)). \quad (2)$$

These transformations form a pseudo-group. They are widely used in control theory [5] – [11].

2. Admissible feedback transformations

Let us determine the feedback transformations preserving a class of oscillation equations.

Denote a space of 2-jets of smooth functions by $J^2(\mathbb{R}^2)$. Let x, y, u be independent variables of these functions. Then the canonical coordinates of this space are

$$u, x, y, y_u, y_x, y_{uu}, y_{ux}, y_{xx}.$$

In the space considered equation (1) specifies a hyper-surface (See [12]):

$$\mathcal{E}_f = \{y_{xx} + f(y, u) = 0\} \subset J^2(\mathbb{R}^2),$$

thereafter also called a control-parameter-dependent quasi-harmonic oscillator equation.

Let \mathcal{E}_f and \mathcal{E}_g be the equations corresponding to functions f and g accordingly. Suppose that

$$\varphi^{(2)}(\mathcal{E}_f) = \mathcal{E}_g, \quad (3)$$

for some function g , where the prolongation of feedback transformation φ to the space of 2-jets is denoted by $\varphi^{(2)}$. Then (3) is equivalent to:

$$\left(\varphi^{(2)}\right)(y_{xx} + f(y, u)) = \lambda(y_{xx} + g(y, u)),$$

where λ is a function at $J^2(\mathbb{R}^2)$.

Having followed an approach proposed by Sophus Lie [13], consider infinitesimal transformations:

$$\varphi_t : (x, y, u) \mapsto (X_t(x, y), Y_t(x, y), U_t(x, y, u)), \quad (4)$$

instead of generalized point transformations (2). Here $X_t(x, y), Y_t(x, y), U_t(x, y, u)$ are the smooth functions of a parameter t . At $t = 0$ φ_0 is an identical transformation at $J^0(\mathbb{R}^2)$. It means that:

$$X_0(x, y) = x, \quad Y_0(x, y) = y, \quad U_0(x, y, u) = u.$$

Now let us determine the vector fields, such that the translations (4) along its trajectories preserve a class of equations (1):

$$X = A(x, y) \frac{\partial}{\partial x} + B(x, y) \frac{\partial}{\partial y} + C(x, y, u) \frac{\partial}{\partial u}.$$

Thereafter the prolongations of vector fields and infinitesimal transformations in the space of 2-jets are denoted by $X^{(2)}$ and $\varphi_t^{(2)}$, respectively (See [12]). The class of equations operated by infinitesimal transformations is preserved under the following condition analogous to (3):

$$\left(\varphi_t^{(2)}\right)^*(y_{xx} + f(y, u)) = \lambda_t(y_{xx} + g_t(y, u)), \quad (5)$$

where λ_t is a local one parameter family of functions at $J^2(\mathbb{R}^2)$, and $g_t(y, u)$ is a local one parameter family of functions of variables y and u , such that:

$$\lambda_0 = 1, \quad g_0(y, u) = f(y, u). \quad (6)$$

Taking the derivative of (5) with respect to t at $t = 0$ and accounting for (6), we obtain:

$$\begin{aligned} \left. \frac{d}{dt} \right|_{t=0} \left(\left(\varphi_t^{(2)}\right)^*(y_{xx} + f(y, u)) \right) &= \lambda_0 \left. \frac{d}{dt} \right|_{t=0} (y_{xx} + g_t(y, u)) + \\ &+ (y_{xx} + g_0(y, u)) \left. \frac{d\lambda_t}{dt} \right|_{t=0} = \left. \frac{d}{dt} \right|_{t=0} g_t(y, u) + (y_{xx} + f(y, u)) \left. \frac{d\lambda_t}{dt} \right|_{t=0}. \end{aligned} \quad (7)$$

The left side of (7) is a Lie derivative along the vector field $X^{(2)}$ of a function $y_{xx} + f(y, u)$. A restriction of the last equality of (7) on \mathcal{E}_f is:

$$L_{X^{(2)}}(y_{xx} + f(y, u))|_{\mathcal{E}_f} = G(y, u),$$

or

$$X^{(2)}(y_{xx} + f(y, u)) \Big|_{\mathcal{E}_f} = G(y, u), \quad (8)$$

where

$$G(y, u) = f(y, u) \frac{d\lambda_t}{dt} \Big|_{t=0}.$$

Vector equation (8) can be rewritten as a system of scalar linear differential equations with respect to functions A , B and C :

$$\begin{cases} A_{yy} = 0, \\ C_x = 0, \\ C_y = 0, \\ B_{yy} - 2A_{xy} = 0, \\ 2B_{xy} - A_{xx} - 3fA_y = 0, \\ B_{xx} - Cf_u - Bf_y - G(y, u) + fB_y - 2fA_x = 0. \end{cases}$$

These equations must be satisfied identically for arbitrary function f and some function G . The general solution of this system is:

$$A(x, y) = \alpha x + \beta, \quad B(x, y) = \gamma + \frac{1}{2}\alpha y + \delta y, \quad C(x, y, u) = c(u).$$

Here $\alpha, \beta, \gamma, \delta$ are arbitrary constants and $c(u)$ is an arbitrary smooth function. Hence any vector fields preserving a class of equations (1) can be presented as a linear combination of the following basal vector fields with respect to feedback transformations:

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial y}, \quad X_3 = y \frac{\partial}{\partial y}, \quad X_4 = x \frac{\partial}{\partial x}, \quad X_5 = c(u) \frac{\partial}{\partial u}.$$

Local translation groups along these fields can be written as follows:

$$\begin{aligned} \varphi_{1,t} : (x, y, u) &\mapsto (x + t, y, u), & \varphi_{2,t} : (x, y, u) &\mapsto (x, y + t, u), \\ \varphi_{3,t} : (x, y, u) &\mapsto (x, e^t y, u), & \varphi_{4,t} : (x, y, u) &\mapsto (e^t x, y, u), \\ \varphi_{5,t} : (x, y, u) &\mapsto (x, y, U_t(u)). \end{aligned}$$

Let us calculate how the transformations $\varphi_{i,t}$ act on equation (1) and on the function f . Transformation $\varphi_{1,t}$ doesn't change the form of equation (1). Applying transformations $\varphi_{2,t}^{(2)} - \varphi_{5,t}^{(2)}$ to the left side of (1) result in:

$$\begin{aligned} \left(\varphi_{2,t}^{(2)}\right)^* (y_{xx} + f(y, u)) &= y_{xx} + f(y + t, u) \\ \left(\varphi_{3,t}^{(2)}\right)^* (y_{xx} + f(y, u)) &= y_{xx} + e^{-t} f(y, u), \\ \left(\varphi_{4,t}^{(2)}\right)^* (y_{xx} + f(y, u)) &= y_{xx} + e^{\frac{t}{2}} f(y, u), \\ \left(\varphi_{5,t}^{(2)}\right)^* (y_{xx} + f(y, u)) &= y_{xx} + f(y, U_t(u)). \end{aligned}$$

3. Differential invariants of quasi-harmonic oscillator equations

Construct a trivial vector bundle with a base \mathbb{R}^3 :

$$\pi : \mathbb{R}^3 \rightarrow \mathbb{R}, \quad \pi : (y, u, z) \mapsto (y, u).$$

The smooth functions

$$s_f : (y, u) \mapsto (y, u, f(y, u))$$

are the cut sets of this bundle. Transformations $\varphi_{2,t}^{(2)}$ - $\varphi_{5,t}^{(2)}$ form a Lie pseudo-group. Correspondent vector fields are the following:

$$Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = y \frac{\partial}{\partial y}, \quad Y_3 = z \frac{\partial}{\partial z}, \quad Y_4 = H(u) \frac{\partial}{\partial u},$$

where H is an arbitrary smooth function. These vector fields form a Lie algebra thereafter denoted by \mathcal{G} .

The differential invariants of equation (1) are the differential invariants of the Lie pseudo-group generated by a set of vector fields Y_1, Y_2, Y_3, Y_4 . The differential invariants of this group are the differential invariants of quasi-harmonic oscillator equation. Let us calculate it.

Let $J^k(\pi)$ be the space of k -jets of a bundle π and $y, u, z, z_x, z_y, z_{xx}, z_{xy}, z_{yy}, \dots$ be the canonical coordinates at this space. Denote the prolongations of vector fields $Y_i (i = 1, \dots, 4)$ to the space $J^k(\pi)$ by $Y_i^{(k)}$.

Definition 1. *The function J at the space of k -jets $J^k(\pi)$ smooth in its domain and rational with respect to variables z_σ on the fibers of bundle π is called differential invariant of order $\leq k$ of Lie group G if it is a constant on the orbits of prolonged Lie group $G^{(k)}$ (See [14]).*

Having solved the system of differential equations:

$$Y^{(k)}(J) = 0, \tag{9}$$

we determine differential invariants of order $\leq k$ of Lie pseudo-group G for any $Y \in \mathcal{G}$. The prolongations of the vector fields Y_i to the space of 2-jets can be written as:

$$\begin{aligned} Y_1^{(2)} &= Y_1, \\ Y_2^{(2)} &= Y_2 - z_y \frac{\partial}{\partial z_y} - 2z_{yy} \frac{\partial}{\partial z_{yy}} - 2z_{yu} \frac{\partial}{\partial z_{yu}}, \\ Y_3^{(2)} &= Y_3 + z_y \frac{\partial}{\partial z_y} + z_u \frac{\partial}{\partial z_u} + z_{yy} \frac{\partial}{\partial z_{yy}} + z_{yu} \frac{\partial}{\partial z_{yu}} + z_{uu} \frac{\partial}{\partial z_{uu}}, \\ Y_4^{(2)} &= Y_4 - H_u(u) z_u \frac{\partial}{\partial z_u} - H_u(u) z_{yu} \frac{\partial}{\partial z_{yu}} - \left(H_{uu}(u) z_u + 2H_u(u) z_{uu} \right) \frac{\partial}{\partial z_{uu}}. \end{aligned}$$

The system (9) being solved at $k = 2$ gives two basic second-order differential invariants.

Theorem 1. *Functions*

$$J_{21} = \frac{z_{yy}z}{z_y^2}, \quad J_{22} = \frac{z_{yu}z}{z_y z_u}$$

form a complete set of the basic second-order differential invariants, i.e. any other second-order differential invariants are the functions of J_{21} and J_{22} .

The prolongations of vector fields Y_i to the space of 3-jets can be written as:

$$\begin{aligned} Y_1^{(3)} &= Y_1, \\ Y_2^{(3)} &= Y_2^{(2)} - 3z_{yyy} \frac{\partial}{\partial z_{yyy}} - 2z_{yyu} \frac{\partial}{\partial z_{yyu}} - z_{yuu} \frac{\partial}{\partial z_{yuu}}, \\ Y_3^{(3)} &= Y_3^{(2)} + z_{yyy} \frac{\partial}{\partial z_{yyy}} + z_{yyu} \frac{\partial}{\partial z_{yyu}} + z_{yuu} \frac{\partial}{\partial z_{yuu}} + z_{uuu} \frac{\partial}{\partial z_{uuu}}, \\ Y_4^{(3)} &= Y_4^{(2)} - H_u(u) z_{yyu} \frac{\partial}{\partial z_{yyu}} - \left(H_{uu}(u) z_{yu} + 2H_u(u) z_{yuu} \right) \frac{\partial}{\partial z_{yuu}} - \\ &\quad - \left(H_{uuu}(u) z_u + 3H_{uu}(u) z_{uu} + 3H_u(u) z_{uuu} \right) \frac{\partial}{\partial z_{uuu}}. \end{aligned}$$

Resolving system (9) at $k = 3$ we obtain three basic third-order differential invariants.

Theorem 2. *Functions*

$$J_{31} = \frac{z_{yyy}z^2}{z_y^3}, \quad J_{32} = \frac{z_{yyu}z^2}{z_y^2 z_u}, \quad J_{33} = \frac{z_{yuu}z^2}{z_u^2 z_y} - J_{22} \frac{z_{uu}z}{z_u^2}.$$

form a complete set of the basic third-order differential invariants.

Other third-order differential invariants are the functions of these invariants. Similarly to it higher-order invariants can be obtained.

Theorem 3. *Functions*

$$\begin{aligned} J_{41} &= \frac{z_{yyyy}z^3}{z_y^4}, \quad J_{42} = \frac{z_{yyyu}z^3}{z_y^3 z_u}, \\ J_{43} &= \frac{z_{yyuu}z^3}{z_u^2 z_y^2} - J_{32} \frac{z_{uu}z}{z_u^2}, \quad J_{44} = \frac{z_{yuuu}z^3}{z_y z_u^3} - 3J_{33} \frac{z_{uu}z}{z_u^2} - J_{22} \frac{z_{uuu}z^2}{z_u^3}, \end{aligned}$$

and

$$\begin{aligned} J_{51} &= \frac{z_{yyyyy}z^4}{z_y^5}, \quad J_{52} = \frac{z_{yyyuu}z^4}{z_y^4 z_u}, \quad J_{53} = \frac{z_{yyuuu}z^4}{z_u^3 z_y^2} - J_{42} \frac{z_{uu}z}{z_u^2}, \\ J_{54} &= \frac{z_{yyuuu}z^4}{z_u^2 z_y^3} - 3J_{43} \frac{z_{uu}z}{z_u^2} - J_{32} \frac{z_{uuu}z^2}{z_u^3}, \\ J_{55} &= \frac{z_{yuuuu}z^4}{z_y z_u^4} - 3J_{44} \frac{z_{uu}z}{z_u^2} - J_{33} \left(\frac{z_{uu}z}{z_u^2} \right)^2 - J_{33} \frac{z_{uuu}z^2}{z_u^3} - J_{22} \frac{z_{uuuu}z^3}{z_u^4} \end{aligned}$$

form a complete set of the basic fourth- and fifth-order differential invariants, respectively.

4. Invariant differentiations of feedback transformations

Invariant differentiations are used to construct high order differential invariants.

Definition 2. *Operator*

$$\nabla = M \frac{d}{dy} + N \frac{d}{du} \quad (10)$$

is called *G-invariant differentiation* if it commutes with every element of any prolongation of Lie algebra \mathcal{G} , where M and N are the functions on the jet space.

It means that the following diagram

$$\begin{array}{ccc} C^\infty(J^\infty(\pi)) & \xrightarrow{\nabla} & C^\infty(J^\infty(\pi)) \\ \downarrow X^{(\infty)} & & \downarrow X^{(\infty)} \\ C^\infty(J^\infty(\pi)) & \xrightarrow{\nabla} & C^\infty(J^\infty(\pi)) \end{array}$$

is commutative for any vector field $Y^* \in \mathcal{G}^{(\infty)}$. Here

$$\begin{aligned} \frac{d}{dy} &= \frac{\partial}{\partial y} + z_x \frac{\partial}{\partial z} + z_{xx} \frac{\partial}{\partial z_x} + z_{xxx} \frac{\partial}{\partial z_{xx}} + \dots, \\ \frac{d}{du} &= \frac{\partial}{\partial u} + z_u \frac{\partial}{\partial z} + z_{uu} \frac{\partial}{\partial z_u} + z_{uuu} \frac{\partial}{\partial z_{uu}} + \dots \end{aligned}$$

are the operators of total differentiation with respect to the variables y and u (See [15]). Invariant differentiation let us construct new differential invariants based on known ones. Indeed let J be the differential invariant and ∇ be the invariant differentiation. Then

$$Y^*(\nabla(J)) = \nabla(Y^*(J)) = 0$$

for any vector field $Y^* \in \mathcal{G}^{(\infty)}$. Therefore the function $\nabla(J)$ is also the differential invariant.

Theorem 4. *Differential operators*

$$\nabla_1 = \frac{z}{z_y} \frac{d}{dy}, \quad (11)$$

$$\nabla_2 = \frac{z}{z_u} \frac{d}{du} \quad (12)$$

are *G-invariant differentiations*.

PROOF. According to [5] if the functions M and N satisfy the following system of differential equations:

$$\begin{cases} X(M) + \frac{d}{dy} \left(\frac{\partial h}{\partial z_y} \right) M + \frac{d}{du} \left(\frac{\partial h}{\partial z_u} \right) N = 0, \\ X(N) + \frac{d}{dy} \left(\frac{\partial h}{\partial z_y} \right) M + \frac{d}{du} \left(\frac{\partial h}{\partial z_u} \right) N = 0, \end{cases}$$

for any vector field $X \in \mathcal{G}$, then operator (10) is G-invariant differentiation. Here h is an arbitrary function. Having resolved this system restricted on the space of 2-jets for the vector fields Y_1, \dots, Y_4 obtain

$$\begin{aligned} M &= \frac{C_1 z}{z_y}, \\ N &= \frac{C_2 z}{z_u}, \end{aligned}$$

where C_1, C_2 are arbitrary constants. Having assumed, $C_1 = 1, C_2 = 1$ obtain invariant differentiations (11)–(12).

Invariant differentiation is determined accurate within the multiplication by differential invariant.

Consider the following decomposition of the invariant differentiations:

$$[\nabla_1, \nabla_2] = \alpha_1 \nabla_1 + \alpha_2 \nabla_2.$$

Remark 1. Functions α_1 and α_2 are differential invariants.

Thus we have:

$$\begin{aligned} \alpha_1 &= -1 + J_{22}, \\ \alpha_2 &= 1 - J_{22}. \end{aligned}$$

where J_{22} is the second order differential invariant. Applying the constructed invariant differentiations $\nabla_{1,2}$ to the second-, third- and fourth- order differential invariants J_{2i}, J_{3i}, J_{4i} obtain the third-, fourth- and the fifth-order differential invariants. Then the following relations are satisfied:

$$\begin{aligned} \nabla_1(J_{21}) &= J_{21} - 2J_{21}^2 + J_{31}, \\ \nabla_2(J_{21}) &= J_{21} - 2J_{21}J_{22} + J_{32}, \\ \nabla_1(J_{22}) &= J_{22} - J_{21}J_{22} - J_{22}^2 + J_{32}, \\ \nabla_2(J_{22}) &= J_{22} - J_{22}^2 + J_{33}. \end{aligned}$$

$$\begin{aligned}
\nabla_1(J_{31}) &= 2J_{31} - 3J_{21}J_{31} + J_{41}, \\
\nabla_2(J_{31}) &= 2J_{31} - 3J_{22}J_{31} + J_{42}, \\
\nabla_1(J_{32}) &= 2J_{32} - 2J_{21}J_{32} - J_{22}J_{32} + J_{42}, \\
\nabla_2(J_{32}) &= 2J_{32} - 2J_{22}J_{32} + J_{43}, \\
\nabla_1(J_{33}) &= 2J_{33} - 2J_{21}J_{33} - 3J_{22}J_{33} + J_{43}, \\
\nabla_2(J_{33}) &= 2J_{33} - J_{22}J_{33} + J_{44}.
\end{aligned}$$

$$\begin{aligned}
\nabla_1(J_{41}) &= 3J_{41} - 4J_{21}J_{41} + J_{51}, \\
\nabla_2(J_{41}) &= 3J_{41} - 4J_{22}J_{41} + J_{52}, \\
\nabla_1(J_{42}) &= 3J_{42} - 3J_{21}J_{42} - J_{22}J_{42} + J_{52}, \\
\nabla_2(J_{42}) &= 3J_{42} - 3J_{22}J_{42} + J_{53}, \\
\nabla_1(J_{43}) &= 3J_{43} - 2J_{21}J_{43} - 2J_{22}J_{43} - J_{33}J_{32} + J_{53}, \\
\nabla_2(J_{43}) &= 3J_{43} - 2J_{22}J_{43} + J_{54}, \\
\nabla_1(J_{44}) &= 3J_{44} - J_{21}J_{44} - 4J_{22}J_{44} - J_{33}^2 + J_{54}, \\
\nabla_2(J_{44}) &= 3J_{44} - J_{22}J_{44} + J_{55}.
\end{aligned}$$

5. Algebras' of differential invariants dimensions

One can easily see the function z to be the relative differential invariant. Indeed

$$Y_1(z) = 0, \quad Y_2(z) = 0, \quad Y_3(z) = z, \quad Y_4(z) = 0.$$

Hence the hyper-surface $z = 0$ consists of the singular orbits of Lie pseudo-group G . Then the following theorem can be formulated

Theorem 5. *Hyper-surface $z = 0$ divides the space $J^0(\pi)$ into two connected components. Lie pseudo-group G operates transitively at any connected component.*

Let us calculate algebras' of differential invariants of order $\leq k$ dimensions. Consider the connected components with $z > 0$ and $z < 0$ separately.

Let us suppose $z > 0$ first. Fix a point $a \in J^0(\pi)$. As Lie pseudo-group G acts transitively at $J^0(\pi)$ it can be assumed that $a = (0, 1, 0)$ without loss of generality. Let $G_a \subset G$ be the isotropy pseudo-group of point a and $\mathcal{G}_a \subset \mathcal{G}$ be the stabilizer of this point. The stabilizer \mathcal{G}_a is generated by the vector fields Y_2 and Y_4 . Let $\pi_k : J^k(\pi) \rightarrow J^0(\pi)$ be the natural projection, i.e. $\pi_k : (y, u, z, z_y, z_u) \mapsto (y, u, z)$. Denote $N_a^k = \pi_k^{-1}(a) \subset J^k(\pi)$, i.e. N_a is the fiber of projection π_k footing into point a . The transformations from Lie pseudo-group preserve the fiber $N^{(k)}(a)$. Vector fields $Y_2^{(k)}$ and $Y_4^{(k)}$ form a basis of prolonged Lie algebra $\mathcal{G}_a^{(k)}$.

Now suppose that $z < 0$. Fix a point $b \in J^0(\pi)$. As Lie pseudo-group G acts transitively at $J^0(\pi)$ it can be assumed that $b = (0, -1, 0)$ without loss of generality. In this case the result is identical to obtained above.

Theorem 6. *Hyper-surfaces*

$$\begin{cases} z = 0, \\ z_y = 0, \\ z_u = 0 \end{cases}$$

divide the space $J^1(\pi)$ into eight connected components. Lie pseudo-group G operates transitively at any connected component.

Thereby we obtain the description of a control-parameter-dependent quasi-harmonic oscillator equation differential invariants' algebra as follows:

Theorem 7. *Quasi-harmonic oscillation equation differential invariants' algebra is generated by second-order differential invariants J_{21} , J_{22} and invariant differentiations ∇_1 and ∇_2 . This algebra separates regular orbits. Algebra's of differential invariants of order $\leq k$ dimensions are equal to:*

$$\nu_k = \frac{k^2 + k - 2}{2}.$$

6. Equivalence problem

Let us call an equation \mathcal{E}_f *regular*, if

$$dJ_{21}(f) \wedge dJ_{22}(f) \neq 0.$$

Here $J(f)$ — is the value of the differential invariant J on the function $f = f(y, u)$.

In a case of regular equations the coordinates y, u can be replaced by the functions $J_{21}(f)$, $J_{22}(f)$ on \mathbb{R}^2 . Since the functions $J_{31}(f)$, $J_{32}(f)$ и $J_{33}(f)$ are also the functions of y, u , there exists a functional dependence between the functions $J_{31}(f)$, $J_{32}(f)$ и $J_{33}(f)$ and J_{21} , J_{22} :

$$J_{3i}(f) = \Phi_{fi}(J_{21}(f), J_{22}(f)).$$

Theorem 8. *Suppose that the functions f and g are real-analytical. Two regular equations \mathcal{E}_f and \mathcal{E}_g are locally G -equivalent if and only if the functions Φ_{if} and Φ_{ig} identically equal ($i = 1, 2, 3$) and 3-jets of the functions f and g belong to the same connection component.*

PROOF. The necessity follows from the invariance of the theorem conditions with respect to Lie group G . Let us prove sufficiency .

Since $\Phi_{if} \equiv \Phi_{ig}$, then the functions f and g are the solutions of the same differential equation. I.e. 3-jets of these functions lay upon the hyper surface \mathcal{E}_f : $[f]^3, [g]^3 \in \mathcal{E}_f$. Let us extract one point on every jet: $a \in [f]^3$ and $b \in [g]^3$. Since

the jets lay in the same connection component, there exists the transformation of Lie group G , converting point b to point a . Thus $a = b$ and $[f]_a^3 = [g]_b^3$. According to Theorem 7 the algebra of differential invariants is generated by invariants J_{21}, J_{22} and invariant differentiations ∇_1, ∇_2 . Hence we obtain that the infinite jets of functions f and g are equal at point a , i.e. $[f]_a^\infty = [g]_b^\infty$. As these functions are analytical, $f \equiv g$ at some neighborhood of point a .

Thereby there exists the transformation of Lie group G , converting the function g to the function f and thus the equation \mathcal{E}_g — to \mathcal{E}_f .

7. Canonical forms of equations

The following equations specify some of canonical forms resulted from equation of class (1).

Theorem 9. *The equation (1) is locally equivalent to the following equation*

$$\frac{d^2 y}{dx^2} + \frac{b(u)^n}{n^n} y^n = 0$$

with respect to feedback transformations if and only if

$$\begin{aligned} J_{21}(f) &= \frac{n-1}{n}, & J_{22}(f) &= 1, & J_{31}(f) &= \frac{n^2 - 3n + 2}{n^2}, \\ J_{32}(f) &= \frac{n-1}{n}, & J_{33}(f) &= 0 \end{aligned}$$

for some natural number n .

Theorem 10. *The equation (1) is locally equivalent to the following equation*

$$\frac{d^2 y}{dx^2} + \alpha(u)y + \beta(u) = 0$$

if and only if $J_{21}(f) = 0$.

Theorem 11. *The equation (1) is locally equivalent to the following equation*

$$\frac{d^2 y}{dx^2} + \alpha(u)y^2 + \beta(u)y + \gamma(u) = 0$$

if and only if $J_{31}(f) = 0$.

8. Conclusions

The classification problem for a control-parameter-dependent quasi-harmonic oscillation equation with respect to feedback transformations has been solved. The structure of the algebra of differential invariants has been described and the equivalence problem has been solved. Some canonical forms have been specified. The implemented methods are based on jet space theory and contact geometry. Analogous methods of classification were applied in [16]–[19]. The technique developed is applicable for the classification problem of general control-parameter-dependent nonlinear ODEs.

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